Bargaining over an Endogenous Surplus

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Abstract

I study a bargaining model between two players with endogenous probability of recognition and surplus. At each period they can make two types of effort: productive effort, that increases the surplus, and unproductive effort, which affects the probability of being recognized as the proposer. With convex effort cost players increase the surplus for a finite number of periods before ending the game. I characterize how the advantages of each player affect the effort decisions over time. I show that advantages in the unproductive effort affect the provision of both types of effort, but advantages in the productive effort only affect the effort decision regarding the productive effort. Different time preferences only affect productive efforts if the probability of recognition is not persistent, and both types of effort if it is.

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1 Introduction

In many bargaining settings, the surplus that players are dividing and the bargaining power are endogenous. Legislators discuss a bill for several periods, making contributions to improve it. At the same time, they spend resources (getting influence, side negotiations) to gain agenda control on how to divide the benefits of the new bill. Similarly, a firm that negotiates with a supplier to get a customized product might prefer to delay the agreement to obtain a better product. At the same time, both parties try to get better conditions in the contract.

If the surplus and the bargaining power depend on each participant's actions over time, the players might prefer not to reach an immediate agreement. They expect to continue making contributions to the surplus to increase the benefits and have a better bargaining position at a future date. Some natural questions arise in this environment: How many resources should a player spend in contributing to the surplus rather than gaining bargaining power? How many periods should the players contribute to the surplus before deciding it is worth dividing?

This paper contributes to the understanding of endogenous bargaining power and how it interacts with the possibility of making contributions to the surplus. I present a bargaining model that captures the tradeoff and intuitions of the problem. Two players can make productive and unproductive efforts each period before they decide to divide the surplus. The productive effort increases the surplus, and the unproductive effort increases the probability of that player being elected the proposer. I characterize how the advantage of a player of having a lower cost of effort affects how much effort both players make over time. I show that advantages in unproductive effort imply a higher provision of both types of effort. Still, advantages in the productive effort only affect the effort decision regarding the productive effort. Different time preferences only affect productive efforts if the probability of recognition is not persistent, and both types of effort if it is.

I use a dynamic bargaining model in discrete time in which two players simultaneously make the two types of efforts at the beginning of each period. The size of the surplus is increasing in the accumulated number of productive efforts of all previous periods. The probability of being recognized as the proposer in each period is higher for the player who makes more unproductive efforts. The elected proposer player offers a division of the surplus to the other player. The game ends if the offer is accepted, and each player receives the agreed division. If the offer is rejected, the game moves to a new period in which, again, they can make both types of efforts.

The cost of the productive effort is convex in each period. It causes the game to be divided into two phases: the contribution and the agreement. In the contribution phase, the players contribute to increasing the surplus without reaching an agreement to end the game. In the second part, when the surplus is large enough, they reach an agreement and the game ends.

I focus on the effect of the specific advantage of a player. I focus on three cases: (1) one player has lower productive costs than the other; (2) one player has lower unproductive costs than the other; and (3) one player is more patient than the other. Also, I consider the cases when the unproductive effort is persistent and non-persistent.

The main result shows that a player with lower costs of unproductive effort makes a higher productive and unproductive effort, even though she does not have an advantage in productive costs. The intuition is a player with lower unproductive cost anticipates that in equilibrium she will make a higher unproductive effort and therefore has a higher probability of being elected proposer. It implies she gets a more significant fraction of the surplus in equilibrium, inducing her to make more contributions. The opposite is not true: A player with lower productive costs makes the same amount of unproductive effort as the other player. The contributions to the surplus are a sunk cost from the perspective of the unproductive effort. Therefore, both players make the same effort in equilibrium if they have the same costs.

Lastly, both players prefer to stop making contributions and split the surplus in the same period, regardless of having different time preferences. Continuation values of contributing one more time are known for both players. Therefore, no matter who the proposer is, if the current surplus is higher than the sum of both continuation values, the proposer will prefer to split the surplus and get her continuation value plus a prize for being the proposer. I also show that a more patient player makes more productive effort, but they make the same unproductive effort if the recognition probability is not persistent. If the recognition probability is persistent, a more patient player makes more of both types of effort.

This paper relates to the literature on endogenous recognition. Yildirim (2007) presents a multilateral legislative bargaining model in which each player can make effort to increase the probability of being elected proposer in each period. Fong and Deng (2012) and Levy and Razin (2013) present a multilateral bargaining model in which the right to be proposer is sold in an all-pay auction. Board and Zwiebel (2012) present a model in which bargainers compete in a first-price auction for the right to propose first. Houba et al. (2022) present a model in which players can compete for each period for the right to propose, focusing on characterizing the entire set of equilibrium payoffs. These models do not account for the possibility of making contributions to the surplus and focus only on the competition for the right to propose.

Che and Sákovics (2004) study a dynamic holdup problem, in which players can make contributions to the surplus in each period if they have not reached an agreement. They use linear costs of contributions and show if players are patient enough, the holdup problem disappears under Markovian strategies. I focus on the contributions over time rather than on the case in which an agreement is reached in the first period. Also, I consider competition for the probability of recognition.

Ali (2015) presents a multilateral bargaining model in which players compete for the right to be the proposer using an all-pay auction. He also considers agents being able to invest in generating surplus before the competition for recognition, showing that players do not invest in surplus in equilibrium. Baranski (2016) presents a model and an experiment of multilateral negotiations in which players invest in a common project before redistributing the total production value. In my model, the timing is different. The competition and the contribution to the surplus are simultaneous at the beginning of each period. Also, I focus on the interaction of both types of effort depending on the advantage of each player.

Other papers that consider a non-constant surplus are Merlo and Wilson (1995), Eraslan and Merlo (2002), and Ortner (2013). However, they consider an exogenous variation of the surplus instead of endogenously determined by the players.

Outline. The rest of the paper is organized as follows. Section 2 introduces the model, and Section 3 considers a benchmark case with only productive efforts. Section 4 analyzes the model with both types of effort, and Section 5 concludes. All the proofs are in the Appendix.

2 Model

I consider a dynamic bargaining model between two players i and j, where time is discrete and indexed by $t = 0, 1, 2....$ The value of the surplus is endogenous and depends on the efforts each player makes at each period. I denote x^{t-1} the value of the surplus at the beginning of period t, where at $t = 0$, the value of the surplus is $x \geq 0$. Each player is risk-neutral, and they discount the future returns and costs by $\delta_i, \delta_j \in (0, 1)$.

At the beginning of each period, both players simultaneously choose productive efforts (e_i^t, e_j^t), and unproductive efforts $(\varepsilon_i^t, \varepsilon_j^t)$. The productive efforts (e_i^t, e_j^t) increase the value of the surplus to

$$x^t = l\Big(\sum_{\tau=0}^{t-1}(e_i^\tau + e_j^\tau) + e_i^t + e_j^t\Big) + x,$$

where $\sum_{\tau=0}^{t-1}(e_i^\tau + e_j^\tau)$ is the sum of all previous productive efforts and $l(\cdot)$ is a strictly increasing, convex, and two-times-differentiable function. These assumptions imply that the surplus is not decreasing over time and that the increment is anonymous—i.e., it does not depend on who made it—so one unit of effort e_i makes the same contribution as one unit of effort e_j.

After the surplus increases to x^t, one of the players is elected the proposer. The probability of being the proposer for each player depends on unproductive efforts $(\varepsilon_i^t, \varepsilon_j^t)$.

$$p(\varepsilon_i, \varepsilon_j) = \begin{cases} p_i > p_j & \text{if } \varepsilon_i > \varepsilon_j \\ p_i = p_j & \text{if } \varepsilon_i = \varepsilon_j \\ p_i < p_j & \text{if } \varepsilon_i < \varepsilon_j, \end{cases}$$

where $\frac{\partial p_i(\varepsilon_i,\varepsilon_j)}{\partial \varepsilon_i} > 0$ and $\frac{\partial^2 p_i(\varepsilon_i,\varepsilon_j)}{\partial (\varepsilon_i)^2} < 0$. It means that the player who exerts more unproductive effort is elected the proposer with a higher probability. If both players make the same amount of effort, the probability of being elected proposer is 1/2 for each player. Note that the probability of being recognized as the proposer only depends on the effort made in the current period t. I consider the persistent recognition case in Section 4.

The player elected proposer makes an offer to the other player. An offer is an allocation $\mathcal{S} = \{\mathcal{S}_i, \mathcal{S}_j\} \in \mathbb{S}$ of the surplus to her opponent and herself. If the responder accepts the proposed allocation, the game ends, and the payoffs are realized. If the responder rejects the allocation, a new period $t + 1$ starts.

The cost of the productive efforts in period t are $c_{e,i} f(e_i)$ and $c_{e,j} f(e_j)$, where $c_{e,i}$ and $c_{e,j}$ are constants and $f(\cdot)$ is a concave and two-times-differentiable function.

The cost of the unproductive efforts is $c_{\varepsilon,i}\varepsilon_i$ and $c_{\varepsilon,j}\varepsilon_j$, where $c_{\varepsilon,i}$ and $c_{\varepsilon,j}$ are constants.

Strategies and solution concept. I focus on Markovian strategies—i.e., strategies that only depend on payoff-relevant information. In each period t the only payoff-relevant information on previous actions is summarized in the size of the surplus x^{t-1}. The efforts are defined as Markovian strategies, where they only depend in the state variable x^{t-1}. After efforts are made, an action $a_i^t(x^t)$ for all i is

$$a_i^t(x^t) \in \begin{cases} \text{pass or } \mathbb{S} & \text{if player } i \text{ is the proposer,} \\ \text{accept or reject} & \text{if not,} \end{cases}$$

where $\mathbb{S}$ is the set of all feasible allocations of the surplus x.

A strategy for player i is a Markovian strategy $(e_i(x^{t-1}), \varepsilon_i(x^{t-1}))$ for the effort and a sequence of actions $a_i^t(x^t)$. A strategy profile is a Markovian subgame perfect equilibrium (MSPE) if it is a Markovian perfect equilibrium in each period t. The solution concept is MSPE.

3 Benchmark: Only productive effort

I first analyze a benchmark model in which players only make productive efforts. The probabilities of being elected proposer are exogenous: p_i and $p_j = 1 - p_i$.

If player i is the proposer, player j rejects any offer lower than her continuation value $\delta_j V_j^t(x^t)$. Thus, if proposer i wants to end the game she needs to offer at least j's continuation value. Optimality requires that the offer is exactly the continuation value. The value of i's payoff is $x^t - \delta_j V_j^t(x^t)$ and j's value is $\delta_j V_j^t(x^t)$.

In period t, the proposer prefers to end the game and splits the surplus if and only if

$$x^t - \delta_j V_j^t(x^t) \geq \delta_i V_i^t(x^t).$$

The above condition means that they reach an agreement if the remaining surplus after she pays the rival player's continuation value (to be indifferent) is equal to or larger than her continuation value. Within a period, players are playing a subgame perfect equilibrium (SPE), and the proposer does not consider the sunk cost of the effort made in the current period. This condition is the same for both players, and it can be written as

$$S^t \equiv x^t - \delta_i V_i^t(x^t) - \delta_j V_j^t(x^t) \geq 0. \tag{1}$$

Value S^t is called *prize* and represents the extra amount over her continuation value the proposer will receive. It is the source of bargaining since players are actually bargaining over who gets it. If the prize is equal to or larger than zero after efforts are made—i.e., condition (1) is satisfied—both players agree to end the game and divide the surplus.

The expected value for player i if the efforts are (e_i^t, e_j^t) and if the game ends is $p_i S^t + \delta_i V_i^t(x^t) - c_i(e_i^t)$. If under the same efforts, condition (1) is not satisfied the expected value is $\delta_i V_i(x^t) - c_i(e_i^t)$. It implies that the optimal expected values at the beginning of period t for players i and j are given by the following maximization problem.

$$\begin{aligned}
V_i(x^{t-1}) &= \max_{e_i^t} \left\{ \max \left\{ \delta_i V_i(x^t) - c_i(e_i^t),\ \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t) \right\} \right\} \\
V_j(x^{t-1}) &= \max_{e_j^t} \left\{ \max \left\{ \delta_j V_j(x^t) - c_j(e_j^t),\ \delta_j V_j^t(x^t) + p_j S^t - c_j(e_j^t) \right\} \right\} \\
&\text{Subject to: } x^t = l(l^{-1}(x^{t-1}) + e_i^t + e_j^t).
\end{aligned} \tag{2}$$

Lemma 1 *The continuation value $V_i(x)$ for each player exists and is unique. Furthermore, $V_i(x)$ is increasing, concave, and differentiable.*

Since $S^t = x^t - \delta_i V_i^t(x^t) - \delta_j V_j^t(x^t)$, equation (2) can be seen as a system of functional equations for functions $V_i^t(x)$ and $V_j^t(x)$. These functions are unique, strictly increasing, and concave in x and differentiable. The uniqueness is an application of the contracting mapping theorem and is strictly increasing and concave because x is strictly increasing and concave. It is differentiable because x and $c(\cdot)$ are differentiable functions.

The game ends when, for optimal efforts, $S(x^t) \geq 0$. It implies $V_i(x^{t-1}) = \delta_i V_i(x^t) - c_i(e_i^t)$ until the last period, in which $V_i(x^{t-1}) = \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t)$.

Lemma 2 *Players will reach an agreement in finite time.*

Lemma 2 says that the condition $x^t - \delta_i V_i(x^t) - \delta_j V_j(x^t) \geq 0$ is satisfied for a finite t. Therefore, the game eventually ends, and players do not contribute forever. Intuitively, if the optimal path of effort is different from zero and gives $S(x^t) < 0$ for all t, the players are facing a path of infinite costs. If that is the case, it is optimal to deviate to a constant path of null efforts; then, the game becomes a Rubinstein (1982) bargaining game in which the surplus does not increase, and the discounted sum of continuation values is equal to the discounted value of the surplus and then $S(x) > 0$. It means that the minimum payoff for each player comes for exerting null efforts and playing a Rubinstein game. Therefore, the game always ends at some finite t.

The above result does not imply that the game always ends at period $t > 0$. If the value of the surplus is big enough at the beginning of the game, then the game ends at $t = 0$.

The game can be understood as a two-phase game:

1. *Construction phase:* Players do not want to end the game, because they prefer to increase the surplus.

2. *Ending phase:* Both players want to end the game, because waiting an additional period does not report net benefits.

3.1 Optimal efforts and the dynamic of the game

Denote by $t + s$ the time the game ends. Note that s is a function of the continuation efforts path. Since the game will end at $t + s$, the value $\delta_i V_i(x^t) - c_i(e_i^t)$ can be written as $\delta_i^s(\delta_i V_i(x^{t+s}) + p_i S^{t+s}) - \sum_{\tau=0}^{s} \delta^\tau c_i(e_i^{t+\tau})$, where the time s and the optimal sequence $\{e_i^\tau, e_j^\tau\}_{\tau=t+1}^{s}$ are functions of the efforts made at t. Then, the first-order conditions can be written as

$$\text{For } S(x^t) \geq 0: \qquad \frac{\partial x^t}{\partial e_i^t}\left[\delta_i \frac{\partial V_i^t(x^t)}{\partial x^t} + p_i \frac{\partial S^t}{\partial x^t}\right] = \frac{\partial c_i(e_i^t)}{\partial e_i^t}$$

$$\text{For } S(x^t) < 0: \qquad \frac{\partial x^{t+s}}{\partial e_i^t}\left[\delta_i^s \frac{\partial V_i^{t+s}(x^{t+s})}{\partial x^{t+s}} + p_i \frac{\partial S(x^{t+s})}{\partial x^{t+s}}\right] = \sum_{\tau=0}^{t+s} \delta_i^\tau \frac{\partial c_i(e_i^{t+\tau})}{\partial e_i^t}.$$

The first-order condition (FOC) for the case $S(x^t) \geq 0$ shows that the optimal condition is to exert effort until the marginal cost is equal to the marginal benefit. The marginal benefit is composed of an increment in the expected prize and an increment in the continuation value. The first-order condition for the case $S(x^t) < 0$ represents the same idea: It is optimal to exert effort until the discounted marginal benefit is equal to the discounted path of costs until the game ends.

Lemma 3 *The sequence of equilibrium efforts* $\{e_i, e_j\}_{t=0}^{\infty}$ *is unique.*

The intuition of Lemma 3 can be seen for the FOCs: The left-hand side is a decreasing function of effort and the right-hand side is increasing in effort.

Lemma 4 *The optimal sequence of effort for each player is decreasing over time. Furthermore, each effort* $e_i(x)$ *is a decreasing and convex function of* x*, where* x *is the size of the surplus at the beginning of the period.*

For the intuition of Lemma 4, consider a best-response sequence $\{e_j\}$ for player j. Then the game can be understood as being to increment the value of x from x^t to x^{t+s} at the lower cost, both in time delay δ_i and $c_i(\cdot)$. Then, if x is *small*, it is less expensive to make increments given the concavity of x. If x is *large*, it is more expensive because to make the same increments as before the effort must be larger. It is optimal to choose a decreasing path of equilibrium effort.

The main result is the interaction between both players and how they use their advantages.

Proposition 1 *The following equilibrium results hold:*

1. *If* $c_i = c_j$, $\delta_i > \delta_j$ *and* $p_i = p_j$ *the efforts are* $e_i(x) > e_j(x)$.

2. *If* $c_i < c_j$, $\delta_i = \delta_j$ *and* $p_i = p_j$ *the efforts are* $e_i(x) > e_j(x)$.

3. *If* $c_i = c_j$, $\delta_i = \delta_j$ *and* $p_i > p_j$ *the efforts are* $e_i(x) > e_j(x)$.

Proposition 1 compares the equilibrium efforts given conditions on the cost, discount factors, and recognition probabilities. If the cost function is the same for both players and they have the same chances of being elected proposer, the more patient player exerts more effort in equilibrium. The future benefits are more important for the more patient player. Her continuation value will be larger than her competitor's continuation value. Considering that the continuation value is the minimum amount the player must receive to agree to finish the game, the patient player has more incentives to exert more effort and ensure a larger payment for herself. Also, she will have a lower adjusted marginal cost.

The second result says that if both players are equally patient and have the same recognition probabilities, but player i is more efficient (lower costs), player i exerts more effort on the equilibrium path. The reason is that player i can increase the continuation value at a lower cost compared with player j. It means that she will get a larger minimum payment for herself. Note that these results also apply for the periods before the game ends.

The last result says that if the players are equally patient and efficient, but one has a larger recognition probability, she will contribute more in equilibrium. Since her chances of being the proposer are larger, she expects to get a larger share of the surplus; then, it is optimal to contribute more in equilibrium.

4 Productive and unproductive effort

Now I consider the general model. As in the benchmark case of only productive effort, there is a take-it-or-leave-it offer at the end of each period. After the proposer is chosen, the subgame is the same as the benchmark model. The player who is elected proposer in period t proposes an allocation the other player agrees with if her payment is larger than her continuation value. It means:

$$S(x^t) \equiv x^t - \delta_i V_i^t(x^t) - \delta_j V_j^t(x^t) \geq 0.$$

This condition is similar to (1). The only difference is that in this case, $V_i^t(x^t)$ is the optimal continuation value for both efforts. The same results regarding the two phases of the game hold in this model, which implies that the game can be separated into the *construction phase* and the *ending phase*.

The expected value for player i if the efforts are $(e_i^t, \varepsilon_i^t, e_j^t, \varepsilon_j^t)$ and the game ends is $p_i S(x^t) + \delta_i V_i^t(x^t) - c_i(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)$. If, under the same efforts, condition $S(x^t) \geq 0$ is not satisfied, the expected value is $\delta_i V_i(x^t) - c_i(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)$. It implies that the optimal expected values at the beginning of period t for player i and j are given by the following maximization problem.

$$
\begin{aligned}
V_i(x^{t-1}) &= \max_{e_i^t,\varepsilon_i^t}\left\{ \max\left\{\delta_i V_i(x^t) - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t,\ \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t\right\}\right\} \\
V_j(x^{t-1}) &= \max_{e_j^t,\varepsilon_i^t}\left\{ \max\left\{\delta_j V_j(x^t) - c_j(e_j^t) - c_{\varepsilon,i}\varepsilon_i^t,\ \delta_j V_j^t(x^t) + p_j S^t - c_j(e_j^t) - c_{\varepsilon,j}\varepsilon_j^t\right\}\right\} \quad (3)\\
&\text{Subject to: } x^t = l(l^{-1}(x^{t-1}) + e_i^t + e_j^t).
\end{aligned}
$$

The same result for $V_i(x)$ in Section 3 applies. The game ends in finite time, and at the beginning of the game there will be a *construction phase* in which $V_i(x^{t-1}) = \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t$. In the last period of the game (*ending phase*), the optimal value is $V_i(x^{t-1}) = \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t$.

In the *construction phase* the optimal unproductive effort is zero because both players know the surplus will not be divided, and thus who is the proposer is not relevant. In the last period, being the proposer becomes valuable, and both players make positive unproductive efforts given by the condition:

$$
S^{t+1}\frac{\partial p_i(\varepsilon_i^t, \varepsilon_j^t)}{\partial \varepsilon_i^t} = c_{\varepsilon,i}(\varepsilon_i^t). \quad (4)
$$

The optimal unproductive effort ε_i is given by the amount of effort such that the marginal cost is equal to the marginal benefit. The marginal benefit is measured as the marginal increment of the expected prize given a change in the probability of winning it. The only objective of the unproductive effort is to improve the chance of winning the prize. The following results explain the advantages of the heterogeneity of both players.

Proposition 2 *Consider t^* to be the last period of the game. Then the following equilibrium results hold:*

1. *If $c_{\varepsilon,i} < c_{\varepsilon,j}$, $c_{e,i} = c_{e,j}$ and $\delta_i = \delta_j$ efforts are $\varepsilon_i^t = \varepsilon_j^t = 0$ for all $t < t^*$, $\varepsilon_i^t > \varepsilon_j^t > 0$ for $t = t^*$ and $e_i^t > e_j^t$ for all $t \le t^*$.*

2. *If $c_{\varepsilon,i} = c_{\varepsilon,j}$, $c_{e,i} = c_{e,j}$ and $\delta_i > \delta_j$ efforts are $\varepsilon_i^t = \varepsilon_j^t = 0$ for all $t < t^*$, $\varepsilon_i^t = \varepsilon_j^t > 0$ for $t = t^*$ and $e_i^t > e_j^t$ for all $t \le t^*$.*

3. *If $c_{\varepsilon,i} = c_{\varepsilon,j}$, $c_{e,i} < c_{e,j}$ and $\delta_i = \delta_j$ efforts are $\varepsilon_i^t = \varepsilon_j^t = 0$ for all $t < t^*$, $\varepsilon_i^t = \varepsilon_j^t > 0$ for $t = t^*$ and $e_i^t > e_j^t$ for all $t \le t^*$.*

The first result of the above Proposition explains that the player who has a comparative advantage in unproductive cost will exert more unproductive effort and have a larger recognition probability. This advantage implies a larger contribution to the joint surplus, even though they are symmetric in other elements. Since the recognition probability is larger, the probability of winning the prize is larger. Hence, she contributes more because she can get a larger share of the surplus.

The second result shows that differences in the discount factor do not play any role in unproductive effort decisions. It is because the players exert unproductive effort only when $S^t \geq 0$, and there is no discount at that moment. Finally, the last result shows that if the only difference is given by the cost of the productive effort, it only affects the productive effort. Since there are no other differences, both players made the same unproductive effort, and the recognition probability is the same for both.

The unproductive effort is null until the condition $S(x) \geq 0$ is satisfied because the objective of that effort is to increase the chance of winning the prize, so the only moment in which the players want to increase their probability of being recognized the proposer is in the period in which they will decide the shares of the surplus, which means at the end of the game

4.1 Persistent effort

In many settings, the effort of gaining bargaining power persists over time. If a player makes effort and increases her bargaining power, it is unlikely she will completely lose it in the next period. To include this extension, I modify the model allowing that the probability of recognition $p(h_t)$ depends on the history of unproductive efforts.

The assumptions about the recognition probabilities are $\frac{\partial p_i(h_t)}{\partial \varepsilon_i^s} > 0$ and $\frac{\partial^2 p_i(h_t)}{\partial (\varepsilon_i^s)^2} < 0$ $\forall i, s \leq t$. It means that the effort made in any period is going to increment the recognition probability in future periods. Furthermore, the recognition probabilities are symmetric. This means that $\frac{\partial p_i(h_t)}{\partial \varepsilon_i^s} = -\frac{\partial p_i(h_t)}{\partial \varepsilon_j^s}$ $\forall i, s \leq t$. The rule $p(h_t) = \{p_i(h_t), p_j(h_t)\}$ is given by:

$$p(h_t) = \begin{cases} p_i(h_t) > p_j(h_t) & \text{if } \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_i^\tau > \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_j^\tau \\ p_i(h_t) = p_j(h_t) & \text{if } \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_i^\tau = \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_j^\tau \\ p_i(h_t) < p_j(h_t) & \text{if } \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_i^\tau < \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_j^\tau, \end{cases}$$

where $\gamma_\tau \in [0, 1]$ is the weight assigned to the effort made in period τ.

Since the recognition is persistent, the unproductive effort depends on all of the previous history: $\varepsilon_i(h_t)$. A strategy for player i is a Markovian strategy $e_i(x)$ for the productive effort, a sequence of strategies $\{\varepsilon_i(h_t)\}_{t=0}^{t=\infty}$ for the unproductive effort, and a sequence of actions $\{a_i^t(x^{t+1})\}_{t=0}^{t=\infty}$. A strategy profile is a Markovian subgame perfect equilibrium (MSPE) if it is a Markov perfect equilibrium in each period t. The solution concept is MSPE.

As in the previous case, the expected value for player i if the efforts are $(e_i^t, \varepsilon_i^t, e_j^t, \varepsilon_j^t)$ and the game ends if $p_i S(x^t) + \delta_i V_i^t(x^t) - c_i(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)$. If, under the same efforts, condition $S(x^t) \geq 0$ is not satisfied, the expected value is $\delta_i V_i(x^t) - c_i(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)$. The optimal expected values at the beginning of period t for players i and j are given by the following maximization problem.

$$\begin{aligned}
&V_i(x^{t-1}) = \max_{e_i^t, \varepsilon_i^t} \left\{ \max \left\{ \delta_i V_i(x^t) - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t,\ \delta_i V_i^t(x^t) + p_i(h^t)S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t \right\} \right\} \\
&V_j(x^{t-1}) = \max_{e_j^t, \varepsilon_i^t} \left\{ \max \left\{ \delta_j V_j(x^t) - c_j(e_j^t) - c_{\varepsilon,i}\varepsilon_i^t,\ \delta_j V_j^t(x^t) + p_j(h^t)S^t - c_j(e_j^t) - c_{\varepsilon,j}\varepsilon_j^t \right\} \right\} \\
&\text{Subject to: } x^t = l(l^{-1}(x^{t-1}) + e_i^t + e_j^t).
\end{aligned} \tag{5}$$

The same results for $V_i(x)$ apply. The game ends in finite time, and at the beginning of the game there will be a *construction phase* in which $V_i(x^{t-1}) = \delta_i V_i^t(x^t) - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t$, and in the last period of the game (*ending phase*) the optimal value will be $V_i(x^{t-1}) = \delta_i V_i^t(x^t) + p_i(h^t)S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t$.

The same first-order condition gives the path of equilibrium productive effort as in the baseline model, and the same results apply. For the unproductive effort, this case is different from the non-persistent case. In the *construction phase* and in the last period, the optimal unproductive effort is positive because being the proposer is valuable for both players and depends on the complete path of efforts. The optimal conditions are

$$\text{For } S(x^t) \geq 0: \qquad \delta_i \frac{\partial V_i(x^t)}{\partial \varepsilon_i^t} + \left[p_i(h^t) \frac{\partial S(x^t)}{\partial \varepsilon_i^t} + S(x^t) \frac{\partial p_i(h^t)}{\partial \varepsilon_i^t} \right] = c_{\varepsilon,i}$$

$$\begin{aligned}
\text{For } S(x^t) < 0: \qquad & \delta_i^{s+1} \frac{\partial V_i(x^{t+s})}{\partial \varepsilon_i^t} + \delta_i^s \left[p_i(h^{t+s}) \frac{\partial S(x^{t+s})}{\partial \varepsilon_i^t} + S(x^{t+s}) \frac{\partial p_i(h^{t+s})}{\partial \varepsilon_i^t} \right] \\
& = \sum_{\tau=0}^{t+s} \delta_i^\tau c_{,i} \frac{\partial_i^{t+\tau}}{\partial_i^t}.
\end{aligned}$$

The optimal unproductive effort ε_i is given by the effort whereby the marginal cost is equal to the marginal benefit. The marginal benefit is measured as the marginal increment of the continuation value and the expected prize, given a change in the probability of winning it.

Proposition 3 *Consider t^* to be the last period of the game. Then the following equilibrium results hold*

1. *If $c_{\varepsilon,i} < c_{\varepsilon,j}$, $c_{e,i} = c_{e,j}$, and $\delta_i = \delta_j$, efforts are $\varepsilon_i^t > \varepsilon_j^t > 0$ for $t \leq t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

2. *If $c_{\varepsilon,i} = c_{\varepsilon,j}$, $c_{e,i} = c_{e,j}$, and $\delta_i > \delta_j$, efforts are $\varepsilon_i^t > \varepsilon_j^t > 0$ for $t \leq t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

3. *If $c_{\varepsilon,i} = c_{\varepsilon,j}$, $c_{e,i} < c_{e,j}$, and $\delta_i = \delta_j$, efforts are $\varepsilon_i^t = \varepsilon_j^t > 0$ for $t \leq t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

The first result of Proposition 3 shows that having a lower cost of unproductive effort means that the agent exerts more effort to increase recognition probability. Because of that, the player will contribute more to the project. Since the recognition probability is larger, the probability of winning the prize is also larger; thus, the expected share of the surplus is larger.

The second result shows that being more patient increases both productive and unproductive effort. Since unproductive effort has a persistent effect, the more patient player will exert more unproductive effort. She perceives larger benefits and lower adjusted marginal costs because she emphasizes future cost reduction, given more unproductive effort today.

Similar to the unproductive effort, the contribution to the surplus is larger for the more patient player because the increment in the future expected prize (reinforced for a larger recognition probability) and the reduction in future cost are more important for her than for her rival player.

Finally, if the only difference between the players is given by the cost of productive effort, in that case, there will be no difference in unproductive effort, and the recognition probability will be the same for both players. The only difference will be given by more productive effort by the player with the lower cost of productive effort.

An important element of the game is how the persistence of the recognition probability works, The more intuitive way is to assume that the effort made two periods ago is less important than the effort made one period ago: $\frac{\partial p_i(h_t)}{\partial \varepsilon_i^s} \geq \frac{\partial p_i(h_t)}{\partial \varepsilon_i^{s'}}$ $\forall i, s' < s \leq t$. The marginal increment in the recognition probability at time t by the effort made in s is greater than or equal to the effort made in s' with $s' < s$. Under the previous assumption: $\gamma_\tau \geq \gamma_{\tau'}$ $\forall \tau \geq \tau'$.

Under the previous assumption, the dynamics of the unproductive efforts are explained in the following Lemma.

Lemma 5 *Unproductive efforts are increasing over time if γ_t is constant or increasing in t.*

Lemma 5 says if recent unproductive efforts are more important for the recognition probability than efforts more distant in time; both players exert an increasing sequence of unproductive efforts over time until the game is over.

Exerting an increasing amount of productive effort is intuitive. Since the recognition process is persistent, it is optimal to exert a positive amount of effort in each period; otherwise, the player loses the advantage over her rival. The effort is increasing since more recent efforts are more important, and at the beginning of the game, the surplus is low, and thus the increment in expected benefit is lower. In later periods the benefits are larger, and thus there are incentives to gain bargaining power.

In the opposite case, if more recent efforts are less important than efforts at the beginning of the game, the dynamics will depend on the specification of the problem. I consider the particular case in which players are equally patient ($\delta_i = \delta_j$) and the weight of each unproductive effort is discounted by $\gamma \in (0,1)$.

Lemma 6 *If $\delta_i = \delta_j = \delta$, under the natural specification of the recognition probability:*[1]

$$p_i(h_t) = f\left(\sum_m^t \gamma^m \varepsilon_i^m, \sum_m^t \gamma^m \varepsilon_j^m\right)$$

- *If $\gamma > \delta$, ε is increasing over time.*
- *If $\gamma = \delta$, ε is constant over time.*
- *If $\gamma < \delta$, ε is decreasing over time.*

Lemma 6 says that if the *discount factor* for the efforts is larger than the benefits and cost's discount factor, then the unproductive effort is increasing over time. The intuition is that even though efforts at the beginning of the game are more important, later efforts are still important—and since players are not very patient, as time progresses, the value of the future payment increases.

[1] An example of this natural probability function is $p_i(h_t) = \frac{\sum_m^t \gamma^m \varepsilon_i^m}{\sum_m^t \gamma^m \varepsilon_i^m + \sum_m^t \gamma^m \varepsilon_j^m}$.

5 Concluding Remarks

This paper contributes to the literature on bargaining over an endogenous surplus. The model considers contributions to a common project in which the shares of the surplus created cannot be decided beforehand. A player might decide to contribute more and make more effort to gain bargaining power; however, this is costly. The tradeoff is how much effort to allocate to contribute to the surplus and gain bargaining power.

In equilibrium, every player has a veto power given by the possibility of rejecting the proposed allocation of the surplus shares. Even in the absence of a formal mechanism, they can ensure a minimum payment that induces them to contribute to the surplus.

The optimal effort is decreasing on the size of the surplus. Also, a more patient player contributes more than a less patient player. The more patient player has a larger continuation value since future payoffs are more valuable for her. A player with lower costs also contributes more in equilibrium than a player with higher costs.

The player with lower unproductive cost exerts more unproductive effort, and she also exerts more productive effort than the other player. She makes more productive effort because her advantage in recognition probability means that she is more likely to be recognized as the proposer. Thus, she will get a larger expected share of the surplus. It generates incentives for her to contribute more to the surplus. This analysis is true for both persistent and not-persistent recognition cases.

In the case of persistent recognition probability, the more patient player will make more productive and unproductive efforts. Since current efforts affect future benefits and costs, being more patient has advantages for recognition probability. Since the more patient player will have a bigger chance of being elected proposer, she also will exert more productive effort.

If the recognition probability is persistent, in cases in which more recent efforts in the current period are more important than more distant in time efforts. In that case, the unproductive effort increases over time. It is increasing because the surplus is low at the beginning of the game; thus, there is no need to exert high effort because the value of the expected payment is low. However, when the surplus is larger, every extra increment in bargaining power is valuable since players compete for the surplus generated, and thus she makes more effort.

A Appendix

A.1 Proof of Lemma 1:

I am going to prove first that the continuation value $V_i(x)$ for each player exist and is unique. The value of $V_i(x)$ for player i in period t is:

$$V_i(x^{t-1}) = \max_{e_i} \left\{ \max \left\{ \delta_i V_i(x^t),\ p_i[x^t - \delta_j V_j(x^t)] + p_j[\delta_i V_i(x^t)] \right\} - c_i(e_i) \right\}$$
$$= \max_{e_i} \left\{ \max \left\{ \delta_i V_i(x^t) - c_i(e_i),\ p_i[x^t - \delta_j V_j(x^t)] + p_j[\delta_i V_i(x^t)] - c_i(e_i) \right\} \right\}$$

Then,

$$V_i(x^{t-1}) = \max_{e_i} \left\{ \max \left\{ \delta_i V_i(x^t) - c_i(e_i),\ p_j \delta_i V_i(x^t) - c_i(e_i) + p_i[x^t - \delta_j V_j(x^t)] \right\} \right\} \tag{6}$$

In the same way, the value of $V_j(x)$ for player j in period t is:

$$V_j(x^{t-1}) = \max_{e_j} \left\{ \max \left\{ \delta_j V_j(x^t) - c_j(e_j),\ p_i \delta_j V_j(x^t) - c_j(e_j) + p_j[x^t - \delta_i V_i(x^t)] \right\} \right\} \tag{7}$$

Note equation (1) and (2) generates a system of functional equations. Replacing the value of x^{t-1} for x and x^t for y, the system can be written as:

$$V_i(x) = \max_{e_i} \left\{ \max \left\{ \delta_i V_i(y) - c_i(e_i),\ p_j \delta_i V_i(y) - c_i(e_i) + p_i[y - \delta_j V_j(y)] \right\} \right\}$$
$$V_j(x) = \max_{e_j} \left\{ \max \left\{ \delta_j V_j(y) - c_j(e_j),\ p_i \delta_j V_j(y) - c_j(e_j) + p_j[y - \delta_i V_i(y)] \right\} \right\}$$

Subject to: $y = l(l^{-1}(x) + e_i + e_j)$

Considering a fix strategy e and $V_j(\cdot)$, i's problem is:

$$V_i(x) = \max_{y \in [x, \infty)} \left\{ \max \left\{ \delta_i V_i(y) - k_i(x,y),\ \beta_i V_i(y) - z_i(x,y) \right\} \right\}$$

where $e_i = l^{-1}(y) - l^{-1}(x) - e_j$, $\beta_i = p_j \delta_i \in (0,1)$ and $z_i(x,y) = c_i(e_i) - p_i[y - \delta_j V_j(y)]$.

$V_i(\cdot)$ is unique because it is a contraction mapping. First, Blackwell's sufficient conditions hold. Define the operator T by:

$$T(f)(x) = \max_{y\in[x,\infty)} \left\{ \max \left\{ \delta_i f(y) - k_i(x,y),\ \beta_i f(y) - z_i(x,y) \right\} \right\} \quad (8)$$

(a) *Monotonicity* consider $f, g \in B(x)$ where $B(x)$ is the set of continuous and bounded functions. Suppose $f(x) \leq g(x)$ for all x. Then:

$$\begin{aligned}
&\max\left\{\delta_i f(y) - k_i(x,y),\ \beta_i f(y) - z_i(x,y)\right\} \\
&\quad\leq \max\left\{\delta_i g(y) - k_i(x,y),\ \beta_i g(y) - z_i(x,y)\right\} \\
&\max_{y\in[x,\infty)} \left\{ \max \left\{ \delta_i f(y) - k_i(x,y),\ \beta_i f(y) - z_i(x,y) \right\} \right\} \\
&\quad\leq \max_{y\in[x,\infty)} \left\{ \max \left\{ \delta_i g(y) - k_i(x,y),\ \beta_i g(y) - z_i(x,y) \right\} \right\} \\
&T(f)(x) \leq T(g)(x)
\end{aligned}$$

(b) *Discounting*

$$\begin{aligned}
T(f+a)(x) &= \max_{y\in[x,\infty)} \left\{ \max \left\{ \delta_i[f(y)+a] - k_i(x,y),\ \beta_i[f(y)+a] - z_i(x,y) \right\} \right\} \\
&= \max_{y\in[x,\infty)} \left\{ \max \left\{ \left[\delta_i f(y) - k_i(x,y)\right] + \delta_i a,\ \left[\beta_i f(y) - z_i(x,y)\right] + \beta_i a \right\} \right\} \\
&\leq \max_{y\in[x,\infty)} \left\{ \max \left\{ \left[\delta_i f(y) - k_i(x,y)\right] + \delta_i a,\ \left[\beta_i f(y) - z_i(x,y)\right] + \delta_i a \right\} \right\} \\
&= T(f)(x) + \delta_i a
\end{aligned}$$

Then T is a contraction, and using the contraction mapping theorem exists an unique continuous and bonded function that satisfies (3). Then V_i and V_j are unique.

$V_i(x)$ is increasing, concave and differentiable for all i:

- *Increasing:* Consider $x < x'$. Note that,

$$\begin{aligned}
k_i(x,y) &= c_i(l^{-1}(y) - l^{-1}(x) - e_j) \\
z_i(x,y) &= c_i(l^{-1}(y) - l^{-1}(x) - e_j) - p_i(y - \delta_j V_j(y))
\end{aligned}$$

increasing x the same y can be chosen at a lower cost. Then $f(x)$ is increasing. Formally,

$$\begin{aligned}
T(f)(x) &= \max_{y\in[x,\infty)}\left\{\max\left\{\delta_i f(y) - k_i(x,y),\ \beta_i f(y) - z_i(x,y)\right\}\right\}\\
&< \max_{y\in[x,\infty)}\left\{\max\left\{\delta_i f(y) - k_i(x',y),\ \beta_i f(y) - z_i(x',y)\right\}\right\}\\
&= \max_{y\in[x',\infty)}\left\{\max\left\{\delta_i f(y) - k_i(x',y),\ \beta_i f(y) - z_i(x',y)\right\}\right\}\\
&= T(f)(x')
\end{aligned}$$

Then $V_i(x)$ is strictly increasing in x.

- *Concave:* $V_i(x)$ is bounded by the next period x. Since x is concave and $V_i(x)$ is strictly increasing, then $V_i(x)$ is concave.
- *Differentiable:* Since $k_i(\cdot)$ and $z_i(\cdot)$ are continuously differentiable, using Benveniste and Scheinkman (1979)'s theorem then V_i is differentiable.

A.2 Proof of Lemma 2

Suppose $S(x) \geq 0$ is never going to be satisfied. The maximization problem of each agent will be:

$$\max_{e_i}\left\{\lim_{\tau\to\infty}\delta_i^\tau V_i(x^\tau) - c_i(e_i)\right\}$$

First note that $V_i(x) \geq 0$. If player i expect a negative continuation value then she can change her strategy to make effort 0 in each subsequent period. It will ensure her a non negative payoff given by her bargaining power and the value of the surplus. Second, since x is a convex function of the efforts, and $V_i(x)$ is bounded for the expected value of x in the next period, then $V_i(x)$ can not increase faster than x. Then $0 \leq \lim_{\tau\to\infty}\delta_i^\tau V_i(x^\tau) \leq \lim_{\tau\to\infty}\delta_i^\tau x^\tau = 0$ implies $\lim_{\tau\to\infty}\delta_i^\tau V_i(x^\tau) = 0$. The maximization problem then becomes:

$$\max_{e_i}\left\{-c_i(e_i)\right\}$$

and the optimal effort is 0 for both players. Now the game become the usual Rubinstein such that in each period $x^t = \delta_1 V_1(x^t) + \delta_2 V_2(x^t)$ since the value of x will not increase. This is a contradiction.

A.3 Proof of Lemma 3

Fix a sequence $\{e_2\}_{t=0}^{\infty}$.

$$V_i(x^{t-1}) = \max_{e_i} \left\{ \max \left\{ \delta_i V_i(x^t) - c_i(e_i),\ \beta_i V_i(x^t) - z_i(e_i, x^t) \right\} \right\} \tag{9}$$

- $\delta_i V_i(x^t) - c_i(e_i)$ is strictly concave, because $\delta_i V_i(x^t)$ is concave and $-c_i(e_i)$ is strictly concave.

  Note $\lim_{e_i \to 0} \delta_i V_i(x^t) - c_i(e_i) = \delta_i V_i(x^t)$ and $\lim_{e_i \to \infty} \delta_i V_i(x^t) - c_i(e_i) = -\infty$, then the maximum value exists and is unique.

- $\beta_i V_i(x^t) - z_i(e_i, x^t)$ is strictly concave for the relevant values of e_i. $\beta_i V_i(x^t)$ is concave considering $-z_i(e_i, x^t) = -c_i(e_i) + p_i[x^t - \delta_j V_j(x^t)]$ it's clear that $-c_i(e_i)$ is strictly concave and $x^t - \delta_j V_j(x^t)$ is concave for values of e_i such that $x^t - \delta_j V_j(x^t) > 0$.

  Note $\lim_{e_i \to 0} \delta_i V_i(x^t) - c_i(e_i) + p_i[x^t - \delta_j V_j(x^t)] = \delta_i V_i(x^t) + p_i[x^t - \delta_j V_j(x^t)]$ and $\lim_{e_i \to \infty} \delta_i V_i(x^t) - c_i(e_i) + p_i[x^t - \delta_j V_j(x^t)] = -\infty$, then the maximum value exists and is unique.

Then for each fix sequence $\{e_2\}_{t=0}^{\infty}$ (5) has an unique maximizer.

A.4 Proof of Lemma 4

First, $e_i(x)$ is a strict monotone function of x: consider x and $\hat{x}$ such that $x \neq \hat{x}$ and $e_i(x) = e_i(\hat{x})$. Then, since $e_i(\cdot)$ is optimal, $V_i(x) = V_i(\hat{x})$ using optimal strategies. But since $V_i(\cdot)$ is a strict monotone function, it implies $x = \hat{x}$. So, $e_i(\cdot)$ is strict monotone function.

Second, $e_i(x)$ is decreasing and convex: using the optimal condition :

$$\frac{\partial x^t}{\partial e_i(x^{t-1})} \left[p_i \frac{\partial S(x^t)}{\partial x^t} + \delta_i \frac{\partial V_i(x^t)}{\partial x^t} \right] = c_i \frac{\partial f(e_i(x^{t-1}))}{\partial e_i(x^{t-1})}$$

if $S(x^t) \geq 0$, or:

$$\frac{\partial x^t}{\partial e_i(x^{t-1})} \delta_i \frac{\partial V_i(x^t)}{\partial x^t} = c_i \frac{\partial f(e_i(x^{t-1}))}{\partial e_i(x^{t-1})}$$

if $S(x^t) < 0$.

Note the left hand side (LHS) is decreasing in effort and right hand side (RHS) is increasing in effort. Consider $\hat{x}^{t-1}$ such that $\hat{x}^{t-1} < x^{t-1}$, then the LHS curve shifts down (and the slope increases) and then the new equilibrium effort is lower

A.5 Proof of Proposition 1:

Result 1: The first order condition can be iterated and then the optimal condition for effort at t when the game is going to end at period $t+s$ is:

$$\sum_{\tau=s}^{\infty} \delta_i^{\tau} p_i \frac{\partial S^{t+\tau}}{\partial x^{t+\tau}} \frac{\partial x^{t+\tau}}{\partial e_i(x^{t-1})} = \sum_{\tau=0}^{\infty} \delta_i^{\tau} c_i \frac{\partial f(e_i(x^{t+\tau}))}{\partial e_i(x^{t-1})}$$

Previous expression can be written as a function of e_i^t and δ_i for player i (symmetrically for player j)

$$LHS_i(e_i^t, \delta_i) = RHS_i(e_i^t, \delta_i)$$

Note that for any value of e $LHS_i(\cdot, \delta_i)$ is larger than $LHS_j(\cdot, \delta_j)$ since $\delta_i > \delta_j$. On the other hand, $RHS_i(\cdot, \delta_i)$ is lower than $RHS_j(\cdot, \delta_j)$ since $\frac{\partial e_i(x^{t+\tau})}{\partial e_i(x^t)}$ is negative, and $\delta_i > \delta_j$.

Since LHS_i is decreasing in e because S is concave, and RHS_i is increasing in e because effort is decreasing and convex, the equilibrium exists and it is unique. Also $e_i(x) > e_j(x)$.

Result 2. The optimal expression can be written as:

$$LHS_i(e_i^t) = RHS_i(e_i^t, c_i)$$

For any value of e $LHS_i(\cdot)$ is equal than $LHS_j(\cdot)$. On the other hand, $RHS_i(\cdot, c_i)$ is lower than $RHS_j(\cdot, c_j)$ because$c_i < c_j$.

Since LHS_i is decreasing and RHS_i is increasing in e, the equilibrium exists and it is unique. Also $e_i(x) > e_j(x)$.

Result 3. The optimal expression can be written as:

$$LHS_i(e_i^t, p_i) = RHS_i(e_i^t)$$

For any value of e $LHS_i(\cdot, p_i)$ is larger than $LHS_j(\cdot, p_j)$ because $p_i > p_j$, and $RHS_i(\cdot)$ is equal than $RHS_j(\cdot)$.

Since LHS_i is decreasing and RHS_i is increasing in e, the equilibrium exists and it is unique. Also $e_i(x) > e_j(x)$.

A.6 Proof of Proposition 2:

Result 1: Optimal expression for ε is:

$$S^{t+1}\frac{\partial p_i(\varepsilon_i^t, \varepsilon_j^t)}{\partial \varepsilon_i^t} = c_{\varepsilon,i}(\varepsilon_i^t)$$

Note the left hand side function of ε is the same for both players. Since it is a decreasing function because p is concave and since the right hand side is constant, then the equilibrium exists, it is unique and , so since $c_{\varepsilon,i} < c_{\varepsilon,j}$ and $\varepsilon_i > \varepsilon_j$. It implies $p_i > p_j$ so using Result 3 of Proposition 1: $e_i(x) > e_j(x)$.

Result 2 and 3: Note that $c_{\varepsilon,i} = c_{\varepsilon,j}$ optimal expression for ε gives the same result for both players. And since optimal expression for e does not play any role in the value of ε then $\varepsilon_i = \varepsilon_j$ and then $p_i = p_j$. So, Result 2 and 3 correspond to Result 1 and 2 of Proposition 5.

A.7 Proof of Proposition 3:

Result 1: Optimal expression for ε is:

$$\sum_{\tau=s}^{\infty} \hat{\delta}_\tau \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} S(x^{t+\tau}) = \sum_{\tau=0}^{\infty} \delta_i^\tau c_{\varepsilon,i} \frac{\partial \varepsilon_i^{t+\tau}}{\partial \varepsilon_i^t}$$

This expression can be written as:

$$LHS_{\varepsilon,i}(\varepsilon_i^t) = RHS_{\varepsilon,i}(\varepsilon_i^t, c_{\varepsilon,i})$$

$LHS_{\varepsilon,i}(\cdot)$ is the same for i and j for ε. $RHS_{\varepsilon,i}(\cdot, c_{\varepsilon,i})$ is lower for player i because $c_{\varepsilon,i} < c_{\varepsilon,j}$. And since $LHS_{\varepsilon,i}$ is decreasing because p is concave and $RHS_{\varepsilon,i}$ is increasing because ε is concave, then the equilibrium exists, it is unique and $\varepsilon_i^t > \varepsilon_j^t$. It implies $p_i(h_t) > p_i(h_t)$, and then $e_i(x) > e_i(x)$ using Result 3 of Proposition 5.

Result 2: Optimal expression for ε can be written as:

$$LHS_{\varepsilon,i}(\varepsilon_i^t) = RHS_{\varepsilon,i}(\varepsilon_i^t, \delta_i)$$

$LHS_{\varepsilon,i}$ is the same function for both players, $RHS_{\varepsilon,i}(\cdot,\delta_i)$ is lower for player i because $\delta_i > \delta_j$ and because $\frac{\partial \varepsilon_i^{t+\tau}}{\partial \varepsilon_i^t}$ is negative. Since $LHS_{\varepsilon,i}$ is decreasing and $RHS_{\varepsilon,i}$ is increasing, the solution exists, is unique and $\varepsilon_i^t > \varepsilon_j^t$. It implies $p_i(h_t) > p_i(h_t)$.

Optimal expression for e can be written as:

$$LHS_{e,i}(e_i^t, \delta_i, p_i) = RHS_{e,i}(e_i^t, \delta_i)$$

For any value of e $LHS_i(\cdot, \delta_i, p_i)$ is larger than $LHS_j(\cdot, \delta_j, p_j)$ since $\delta_i > \delta_j$ and $p_i > p_j$. On the other hand, $RHS_i(\cdot, \delta_i)$ is lower than $RHS_j(\cdot, \delta_j)$ since $\frac{\partial e_i(x^{t+\tau})}{\partial e_i(x^t)}$ is negative, and $\delta_i > \delta_j$.

Since LHS_i is decreasing in e and RHS_i is increasing in e, the equilibrium exists and it is unique. Also $e_i(x) > e_j(x)$.

Result 3: Since there are not comparative advantages that affect ε, then $\varepsilon_i = \varepsilon_j$. It implies $p_i = p_j$. Then using Result 2 of Proposition 5 the effort are $e_i(x) > e_j(x)$.

A.8 Proof of Lemma 5:

Optimal condition for ε is:

$$\sum_{\tau=s}^{\infty} \hat{\delta}_\tau \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} S^{t+1+\tau} = \sum_{\tau=0}^{\infty} \delta_i^\tau c_{\varepsilon,i} \frac{\partial \varepsilon_i^{t+\tau}}{\partial \varepsilon_i^t}$$

$$\sum_{\tau=s}^{\infty} \hat{\delta}_\tau \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} S(x^{t+\tau}) = \sum_{\tau=0}^{\infty} \delta_i^\tau c_{\varepsilon,i} \frac{\partial \varepsilon_i^{t+\tau}}{\partial \varepsilon_i^t}$$

if the current period is t and the game will finish at $t+s$.

Case 1: Recent efforts are more important (γ_t increasing in t): $\frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} < \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t+1}}$. LHS for optimal condition in if the current period is $t+1$ is:

$$
\begin{aligned}
LHS_{t+1} &= \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau-1} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t+1}} S(x^{t+\tau}) \\
&> \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau-1} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t}} S(x^{t+\tau}) \\
&> \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t}} S(x^{t+\tau}) = LHS_t
\end{aligned}
$$

Note that $\hat{\delta}_\tau \leq \hat{\delta}_{\tau'}$ $\forall \tau > \tau'$.

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} > LHS_t$ the value of ε is $\varepsilon_i^{t+1} > \varepsilon_i^t$ $\forall i$.

Case 2: Recent efforts are equally important than later efforts (γ_t increasing in t):

$$
\frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} = \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t+1}}.
$$

LHS for optimal condition in if the current period is $t+1$ is:

$$
\begin{aligned}
LHS_{t+1} &= \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau-1} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t+1}} S(x^{t+\tau}) \\
&= \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau-1} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t}} S(x^{t+\tau}) \\
&> \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t}} S(x^{t+\tau}) = LHS_t
\end{aligned}
$$

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} > LHS_t$ the value of ε is $\varepsilon_i^{t+1} > \varepsilon_i^t$ $\forall i$.

A.9 Proof of Lemma 6:

Note:

$$
\frac{\partial p_i(h_{t+\tau})}{\varepsilon_i^t} = \frac{\partial f(\sum_{m=0}^{t+\tau} \gamma^m \varepsilon_i^m, \sum_{m=0}^{t+\tau} \gamma^m \varepsilon_j^m)}{\partial \sum_{m=0}^{t+\tau} \gamma^m \varepsilon_i^m} \gamma^t = \varphi(h_{t+\tau})\gamma^t
$$

then the optimal condition for ε is:

$$\sum_{\tau=s}^{\infty} \delta^{\tau} \varphi(h_{t+\tau}) \gamma^{t} S(x^{t+\tau}) = \sum_{\tau=0}^{\infty} \delta_i^{\tau} \frac{\partial c_{\varepsilon,i}(\varepsilon_i^{t+\tau})}{\partial \varepsilon_i^t}$$

if the current period is t and the game will finish at $t+s$.

Result 1: $(\gamma > \delta)$ LHS for optimal condition in if the current period is t is:

$$LHS_t = \sum_{\tau=s}^{\infty} \delta^{\tau} \varphi(h_{t+\tau}) \gamma^{t} S(x^{t+\tau}) = \delta \gamma^{t} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau})$$
$$< \gamma^{t+1} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau}) = LHS_{t+1}$$

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} > LHS_t$ the value of ε is $\varepsilon_i^{t+1} > \varepsilon_i^t \ \forall i$.

Result 2: $(\gamma = \delta)$ LHS for optimal condition in if the current period is t is:

$$LHS_t = \sum_{\tau=s}^{\infty} \delta^{\tau} \varphi(h_{t+\tau}) \gamma^{t} S(x^{t+\tau}) = \delta \gamma^{t} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau})$$
$$= \gamma^{t+1} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau}) = LHS_{t+1}$$

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} = LHS_t$ the value of ε is $\varepsilon_i^{t+1} = \varepsilon_i^t \ \forall i$.

Result 3: $(\gamma < \delta)$ LHS for optimal condition in if the current period is t is:

$$LHS_t = \sum_{\tau=s}^{\infty} \delta^{\tau} \varphi(h_{t+\tau}) \gamma^{t} S(x^{t+\tau}) = \delta \gamma^{t} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau})$$
$$> \gamma^{t+1} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau}) = LHS_{t+1}$$

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} > LHS_t$ the value of ε is $\varepsilon_i^{t+1} > \varepsilon_i^t \ \forall i$.

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